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Transitivity of local complementation and switching on graphs

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Abstract

The operations complementation C, local complementation λ_x, and switching σ_x for the vertices x of a finite undirected graph are considered. The operation λ_x complements the subgraph induced by the neighbourhood of x in the given graph, and the switching σ_x changes the neighbourhood of x to its complement vertex set. It is proved that the compositions $\delta_x = \lambda_x C$ (for vertices $x \in D$) generate a transitive group on the graphs with vertex set D, that is, for any two graphs g and h on D, there exists a composition α of operations δ_x such that $h = \alpha(g)$. It is also shown that the compositions $\tau_x = \lambda_x \sigma_x$ (for $x \in D$) generate a transitive group on the graphs.

Keywords: Finite graph; Local complementation; Switching; Transitive group; Composition of operations

1. Introduction

The graphs of this paper will be finite, undirected, without loops and multiple edges, that is, a graph is a pair $g = (D, E)$, where D is a finite set of vertices, and E is a set of edges included in

$$E(D) = \{\{x, y\} \mid x, y \in D,\ x \neq y\}.$$

We also say that the graph g is *on* D.

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We shall consider the following three operations on graphs. The *complement* of a graph $g=(D,E)$ is the graph $C(g)=(D,E(D)\setminus E)$, where the edges are changed to nonedges and vice versa. For a vertex $x\in D$, the *local complement* of g with respect to x, denoted by $\lambda_x(g)$, is the graph, where the subgraph of g induced by the neighbourhood $N_g(x)$ of x is complemented. The *switching* of g at the vertex x, denoted by $\sigma_x(g)$, is the graph obtained from g by changing the edges incident with x to nonedges and vice versa.

We shall prove in Section 4 that the operations complementation and local complementations generate a transitive group on the set of all graphs on a fixed vertex set D, that is, if g and h are graphs on D, then there exists a composition α of the operations C and λ_x $(x\in D)$ such that $h=\alpha(g)$. We prove two improvements of this result, the first of which states that local complementations can be restricted to λ_x for a fixed vertex x. The second improvement states that the operations $\delta_x=\lambda_x C$ (that first complement the graph and then locally complement it with respect to the given vertex) generate a transitive group on the graphs on D. In particular, all graphs can be constructed from the discrete graphs with no edges by using these operations δ_x.

In Section 5, we show that complementation can be expressed in terms of switchings and local complementations with respect to any two vertices. Indeed, we prove that there is such a composition

$$C=\lambda_x\lambda_y\sigma_x\sigma_y\lambda_y\lambda_x\sigma_y\lambda_x\lambda_y\sigma_y\sigma_x\lambda_y\lambda_x$$

of 13 applications of the operations λ_x, λ_y, σ_x and σ_y for $x\neq y$. It follows from this that switchings and local complementations generate a transitive group on the graphs. We also consider the operations $\tau_x=\lambda_x\sigma_x$ that first switch a graph and then locally complement the resulting graph with respect to the same vertex x. We show that the operations τ_x (for $x\in D$) generate a transitive group on the graphs with the vertex set D.

In Section 6, a lower and an upper bound is given for the number of operations C,σ_x,λ_x $(x\in D)$ needed to construct any graph from the discrete graph.

Local complementations in graphs have been studied, for instance, by Bouchet [1] and Fon-der-Flaas [3]. In particular, the following result was proved in these two papers.

Proposition 1. *Let α be a composition of local complementations. If g and $\alpha(g)$ are both trees, then they are isomorphic.*

Similar result holds for switchings. The following proposition was proved by Hage and Harju [5].

Proposition 2. *Let α be a composition of switchings. If g and $\alpha(g)$ are both trees, then they are isomorphic.*

By these two propositions, the operations local complementation and switching are rather restricted when applied alone without the help of complementation or of each other.

Switching of graphs was introduced by Van Lint and Seidel [7] in connection with a problem of equilateral n-tuples of points in elliptic geometry. For further results on switching of graphs, see, for instance [2,4,6].

2. Preliminaries

The cardinality of a finite set X is denoted by $|X|$. For two sets A and B, we denote by $A+B$ their *symmetric difference*, i.e., $A+B=(A\setminus B)\cup(B\setminus A)$.

Let $\mathscr{G}_D$ be the set of all graphs on the set D of vertices. For a graph $g=(D,E)$, the set of edges $E\subseteq E(D)$ of g will be also denoted by E_g. Also, we write simply xy instead of $\{x,y\}\in E(D)$, and a singleton $\{x\}$ is often identified with its sole member x. For two disjoint subsets $A,B\subseteq D$, let $E(A,B)=\{xy \mid x\in A, y\in B\}$ be the set of all pairs between the sets A and B.

For a vertex $x\in D$, let $N_g(x)=\{y \mid xy\in E\}$ be the *neighbourhood* of x in g. A vertex x is *isolated*, if $N_g(x)=\emptyset$, and x is *universal*, if $N_g(x)=D\setminus\{x\}$. Also, let $N'_g(x)=D\setminus(N_g(x)\cup\{x\})$ be the set of non-neighbours of x.

The *discrete graph* on D has no edges, and it is defined by $\mathbf{0}_D=(D,\emptyset)$.

For a subset $A\subseteq D$, let $g[A]=(A,E\cap E(A))$ be the *subgraph* of g *induced* by A. For a vertex $x\in D$, we adopt a shorter notation $g-x=g[D\setminus\{x\}]$. We denote by $g\oplus A$ the graph which is obtained from g by complementing the subgraph $g[A]$ and leaving the rest of g unchanged, that is,

$$g\oplus A=(D,E+E(A)).$$

For subsets $A_i\subseteq D$, for $i=1,2,\ldots,k$, we write $g\oplus A_1\oplus A_2\oplus\cdots\oplus A_k$ for $((\cdots(g\oplus A_1)\oplus A_2)\oplus\cdots)\oplus A_k$.

For disjoint subsets A and B, we let

$$g\oplus[A,B]=(D,E+E(A,B)),$$

that is, $g\oplus[A,B]$ is obtained from g by complementing the connections between the sets A and B.

The following two results is easy to prove.

Lemma 3. *Let $g=(D,E)$ be a graph, and let $A,B\subseteq D$. Then*

(1) $g\oplus A\oplus A=g$,
(2) $g\oplus A\oplus B=g\oplus B\oplus A$.

Lemma 4. *Let $g=(D,E)$ be a graph, and let $A,B\subseteq D$ be disjoint subsets. Then*

(1) $g\oplus(A\cup B)=g\oplus A\oplus B\oplus[A,B]$,
(2) $g\oplus[A,B]=g\oplus A\oplus B\oplus(A\cup B)$.

Let $\mathrm{Sym}(\mathscr{G}_D)$ be the symmetric group on $\mathscr{G}_D$ consisting of all permutations of the set $\mathscr{G}_D$ of the graphs on D. The identity permutation on $\mathscr{G}_D$ is denoted by $\imath$. Let

$\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\} \subseteq \mathrm{Sym}(\mathscr{G}_D)$. Then denote by $\langle \gamma_1, \gamma_2, \ldots, \gamma_n \rangle_D$ as well as by $\langle \Gamma \rangle_D$ the subgroup of $\mathrm{Sym}(\mathscr{G}_D)$ generated by the elements of Γ.

Let $\Gamma \subseteq \mathrm{Sym}(\mathscr{G}_D)$. Two graphs g and h on D are said to be *Γ-equivalent*, if there exists an element $\gamma \in \langle \Gamma \rangle_D$ such that $\gamma(g) = h$. The group $\langle \Gamma \rangle_D$ is called *transitive* (on $\mathscr{G}_D$), if g and h are Γ-equivalent for all graphs $g, h \in \mathscr{G}_D$. In order to show that a subgroup $\langle \Gamma \rangle_D$ of $\mathrm{Sym}(\mathscr{G}_D)$ is transitive it is sufficient to show that every graph is Γ-equivalent to a fixed graph $g_0 \in \mathscr{G}_D$. Indeed, if $\alpha(g) = g_0$ and $\beta(h) = g_0$ for some $\alpha, \beta \in \langle \Gamma \rangle_D$, then $\beta^{-1}\alpha(g) = h$.

We note that $C(g) = g \oplus D$ for the complement of a graph $g = (D, E)$.

The *local complement* of a graph g with respect to a vertex x is the graph

$$\lambda_x(g) = g \oplus N_g(x),$$

where the subgraph induced by the neighbourhood of x is complemented. Let $\Lambda = \{\lambda_x \mid x \in D\}$ be the set of the local complementations on D.

The *switching* of g at a vertex x is defined as the graph $\sigma_x(g)$, which is obtained from g by changing the status of each pair xy, $y \in D \setminus \{x\}$, as an edge, that is,

$$\sigma_x(g) = g \oplus [x, D \setminus \{x\}].$$

Let $\Sigma = \{\sigma_x \mid x \in D\}$ be the set of the switchings on D.

Clearly, $g \oplus \{x\} = g$ for all $g \in \mathscr{G}_D$ and $x \in D$. By Lemma 4(2), $g \oplus [x, D \setminus \{x\}] = g \oplus \{x\} \oplus (D \setminus \{x\}) \oplus D = g \oplus (D \setminus \{x\}) \oplus D$ and thus, by Lemma 3(2),

$$\sigma_x(g) = g \oplus D \oplus (D \setminus \{x\}) \tag{1}$$

and also $\sigma_x(g) = C(g) \oplus (D \setminus \{x\})$, since $C(g) = g \oplus D$.

From the definitions it is clear that $C, \lambda_x, \sigma_x \in \mathrm{Sym}(\mathscr{G}_D)$, for all $x \in D$, and that these permutations have order 2, i.e., $C^2 = \lambda_x^2 = \sigma_x^2 = \iota$, for all x. Furthermore, the group $\langle \Sigma \rangle_D$ of switchings is an abelian subgroup of $\mathrm{Sym}(\mathscr{G}_D)$, i.e., $\sigma_x \sigma_y = \sigma_y \sigma_x$ for all x, y. Also, it is clear that complementation C commutes with the switchings σ_x, i.e., $\sigma_x C = C \sigma_x$.

The groups $\langle \Sigma \rangle_D$ and $\langle \Lambda \rangle_D$ are not transitive on $\mathscr{G}_D$ if $|D| \geqslant 3$. Indeed, the discrete graph $\mathbf{0}_D$ is Σ-equivalent only to the complete bipartite graphs, see [6,2,4], and, obviously, $\mathbf{0}_D$ is Λ-equivalent only to itself.

Example 5. Let $D = \{1, 2, 3\}$, and consider the graph g of Fig. 1. The graphs that are Σ-equivalent to g are given in Fig. 1. Similarly, the graphs that are Λ-equivalent to g are given in Fig. 2. As is easily seen from Fig. 1, the group $\langle C, \Sigma \rangle_D$, generated by

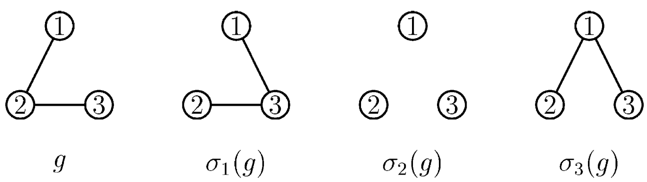

Fig. 1. Switchings of the graph g.

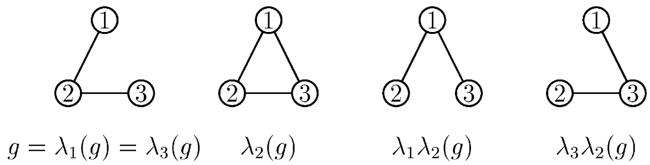

Fig. 2. Local complementations of the graph g.

the switchings together with the complementation, is transitive on $D=\{1,2,3\}$. This property fails for larger vertex sets, since the discrete graph is (C,Σ)-equivalent to a complete bipartite graph or a complement of a complete bipartite graph (and the path P_4 of four vertices is neither of these types). As we shall see in the following sections, the groups $\langle C,\Lambda\rangle_D$ and $\langle \Sigma,\Lambda\rangle_D$ are both transitive on $\mathscr{G}_D$.

For a subset $A=\{x_1,x_2,\ldots,x_n\}$ of vertices of a graph g, we denote by

$$\sigma_A=\sigma_{x_n}\sigma_{x_{n-1}}\ldots\sigma_{x_1} \tag{2}$$

the composition of the switchings σ_x, $x\in A$. Since $\langle\Sigma\rangle_D$ is abelian, σ_A is well defined, i.e., it is independent of the order of the elements x_i in (2). It is well known, see [2,4,6], that for all subsets $A,B\subseteq D$,

$$\sigma_B\sigma_A=\sigma_{A+B} \quad \text{and} \quad \sigma_A(g)=\sigma_{D\setminus A}(g). \tag{3}$$

3. Basic results

We begin with a simple result for complementation and switchings.

Lemma 6. *Let $g=(D,E)$ be a graph, and $A\subseteq D$. Then*

$$C(g\oplus A)=C(g)\oplus A \quad \textit{and} \quad \sigma_x(g\oplus A)=\sigma_x(g)\oplus A. \tag{4}$$

Proof. Using the commutativity result in Lemma 3(2), we obtain

$$C(g)\oplus A=(g\oplus D)\oplus A=(g\oplus A)\oplus D=C(g\oplus A)$$

and, by (1) and Lemma 3(2),

$$\begin{aligned}\sigma_x(g)\oplus A&=g\oplus D\oplus(D\setminus\{x\})\oplus A\\&=(g\oplus A)\oplus D\oplus(D\setminus\{x\})\\&=\sigma_x(g\oplus A),\end{aligned}$$

as required. □

In order to shorten and to have more illustrative proofs, we consider representations, or diagrams, of graphs by vertices and induced subgraphs as explained in the following.

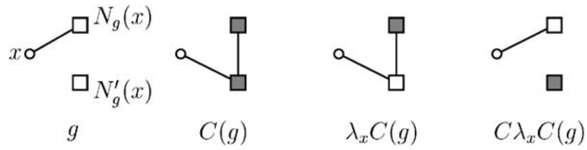

Fig. 3. Derivation of $C\lambda_x C(g)$.

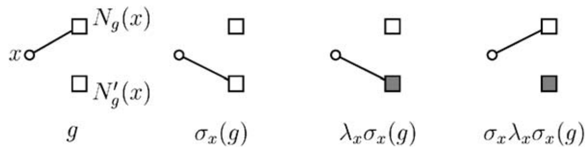

Fig. 4. Derivation of $\sigma_x\lambda_x\sigma_x(g)$.

Let $g=(D,E)$ be a graph. Moreover, let $A=\{x_1,x_2,\ldots,x_k\}\subseteq D$ be a set of its vertices and $S=\{N_1,N_2,\ldots,N_m\}$ a set of subsets of D such that

(i) $A,N_1,N_2,\ldots,N_m$ form a partition of D, and
(ii) for each $x\in A$ and $N\in S$, either $N\subseteq N_g(x)$ or $N\subseteq N'_g(x)$.

(Recall that $N'_g(x)=D\setminus(N_g(x)\cup\{x\})$.) In the *diagram* (A,S) for g by A and S, the vertices $x\in A$ are given as circles, and the sets $N\in S$ (or the subgraphs $g[N]$) are given as squares. We draw a line connecting the vertices $x,y\in A$ if $xy\in E$, and a line connecting a vertex $x\in A$ and a set $N\in S$, if $N\subseteq N_g(x)$. Initially there are no lines between the squares N_i and N_j. Relative to a given diagram for g, diagrams of other graphs can be drawn as follows, as long as the induced subgraphs $g[N]$, for $N\in S$, and the sets of edges connecting N and M, for $N,M\in S$, are either unchanged or they are complemented. If the subgraph $g[N]$ is complemented, then it is represented by a black square, and if the connections between M and N are complemented, then a line is drawn between the two squares.

Example 7. Figs. 3 and 4 show how to derive the identity

$$C\lambda_x C=\sigma_x\lambda_x\sigma_x \tag{5}$$

in the group $\langle C,\Sigma,\Lambda\rangle_D$. Let $g=(D,E)$ be any graph on D. We have chosen the diagram (A,S) for g, where $A=\{x\}$ and $S=\{N_g(x),N'_g(x)\}$. Notice that, in the first diagram in Fig. 3, the edges between the vertices in the sets $N_g(x)$ and $N'_g(x)$ have not been indicated. (In the second diagram the connections between these sets have been complemented.)

As another example, we shall consider the compositions

$$\eta_x=\lambda_x\sigma_x\lambda_x$$

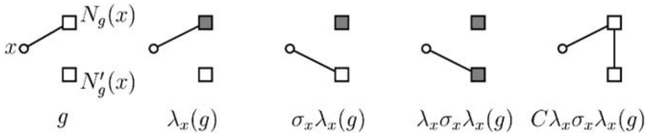

Fig. 5. Derivation of $C\eta_x(g) = C\lambda_x\sigma_x\lambda_x(g)$.

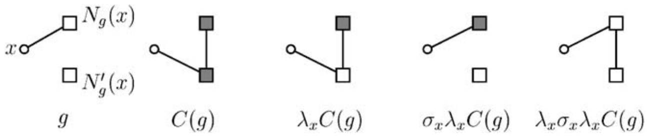

Fig. 6. Derivation of $\eta_xC(g) = \lambda_x\sigma_x\lambda_xC(g)$.

for vertices x. These operations are used in Section 6, where we count the operations needed to generate a graph from the discrete graph. We show that the operations η_x commute with the complementation C.

Lemma 8. *For all elements $x \in D$, $C\eta_x = \eta_x C$. Moreover, for all graphs $g = (D, E)$, we have $C\eta_x(g) = g \oplus [N_g(x), N'_g(x)]$.*

Proof. Let $g = (D, E)$ be any graph on D. We choose the diagram (A, S) for g, where $A = \{x\}$ and $S = \{N_g(x), N'_g(x)\}$. With respect to this diagram, the diagram for the graph $C\eta_x(g)$ is derived in Fig. 5, and the diagram for $\eta_xC(g)$ is derived in Fig. 6. Therefore, $C\eta_x(g) = \eta_xC(g)$, and also the second claim is clear from Fig. 5 (and Fig. 6). $\square$

4. Complementation with local complementation

In this section, we shall show that the operations of complementation and local complementations generate a transitive subgroup on $\mathscr{G}_D$, that is, for all graphs $g, h \in \mathscr{G}_D$, there exists a composition $\alpha = \alpha_n\alpha_{n-1}\ldots\alpha_1$, where $\alpha_i = C$ or $\alpha_i \in \Lambda$ for each i, such that $\alpha(g) = h$.

We begin with a result for graphs with an isolated vertex.

Lemma 9. *Let $g \in \mathscr{G}_D$ be a graph with an isolated vertex x. Then x is isolated in $\lambda_z(g)$ and $\lambda_z(g) - x = \lambda_z(g - x)$ for all z with $z \neq x$. Also, x is isolated in $C\lambda_xC(g)$, and $C\lambda_xC(g) - x = C(g - x)$.*

Proof. The first claim is obvious, since $x \notin N_g(z)$. In general, we have $C\lambda_xC(g) = g \oplus N'_g(x)$; see Fig. 3 in Example 7. Now, $N'_g(x) = D\setminus\{x\}$, and hence $C\lambda_xC(g) = g \oplus (D\setminus\{x\})$. Therefore, x is isolated in $C\lambda_xC(g)$, and then also $C\lambda_xC(g) - x = C(g - x)$. $\square$

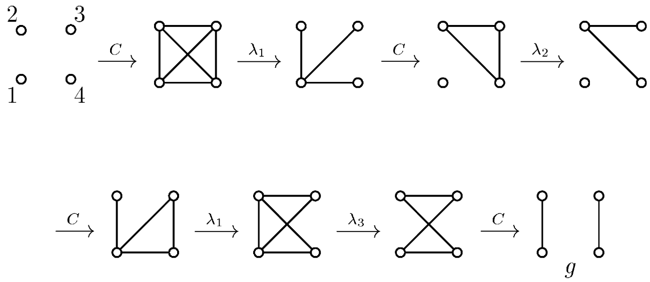

Fig. 7. Case $|D|=4$ for $\langle C,\Lambda\rangle_D$.

We now prove our first transitivity result.

Theorem 10. *The group $\langle C,\Lambda\rangle_D$ generated by complementation and local complementations is transitive on $\mathscr{G}_D$.*

Proof. The proof is by induction on the cardinality of the set of vertices D. If $|D|\leqslant 2$, then the group $\langle C\rangle_D$ is already transitive on $\mathscr{G}_D$. Assume now that the claim holds for sets of vertices of cardinality $n-1$, and let $|D|=n$. Let $g\in\mathscr{G}_D$. We show that there exists an operation $\varphi\in\langle C,\Lambda\rangle_D$ such that $\varphi(g)=\mathbf{0}_D$. This will prove the claim.

We show first that there exists an operation $\beta\in\langle C,\Lambda\rangle_D$ such that $\beta(g)$ contains an isolated vertex x.

Let $z\in D$, and denote $h=g-z$. By the induction hypothesis, there exists an operation $\gamma\in\langle C,\Lambda\rangle_D$ such that $\gamma(h)=\mathbf{0}_{D\setminus\{z\}}$. Clearly, $\gamma(h)=\gamma(g)-z$, since h is an induced subgraph of g. If z is not universal in $\gamma(g)$, then each vertex $x\notin N_{\gamma(g)}(z)$ is isolated in $\gamma(g)$, and, if z is universal in $\gamma(g)$, then $x=z$ is isolated in $C\gamma(g)$.

For the proof of the theorem, we can assume that x is isolated already in g. Let h and γ be as above with $\gamma=\gamma_nC\gamma_{n-1}C\ldots C\gamma_1$, where $\gamma_i\in\langle\Lambda\rangle_D$ for each i (possibly $\gamma_i=\imath$ for $i=1$ or n). By Lemma 9, $\alpha(g)=\mathbf{0}_D$, where $\alpha=\gamma_nC\lambda_xC\gamma_{n-1}C\lambda_xC\ldots C\lambda_xC\gamma_1$ is obtained from the presentation $\gamma_nC\gamma_{n-1}C\ldots C\gamma_1$ of γ by replacing each occurrence of C by the composite operation $C\lambda_xC$ (giving $\gamma_n(C\lambda_xC)\gamma_{n-1}(C\lambda_xC)\ldots(C\lambda_xC)\gamma_1$). □

Corollary 11. *Every graph can be constructed from the discrete graphs by using the operations of complementation and local complementation.*

Example 12. For $|D|=4$ it can be shown that *one needs eight operations* from $\Lambda\cup\{C\}$ to construct the graph g with two independent edges from $\mathbf{0}_D$, see Fig. 7. □

For each vertex $x\in D$, define the operation

$$\delta_x=\lambda_xC \tag{6}$$

(see Fig. 8). Note that now $\delta_x^{-1}=C\lambda_x$. Let also $\Delta=\{\delta_x\mid x\in D\}$.

Lemma 13. *Let $g\in\mathscr{G}_D$ be a graph with an isolated vertex x. Then x is isolated in $\delta_z\delta_x^{-1}(g)$, and, moreover, $\delta_z\delta_x^{-1}(g)=\lambda_z(g)$ for all z with $z\neq x$. Also, x is isolated in $\delta_x^2(g)$, and we have $\delta_x^2(g)-x=C(g-x)$.*

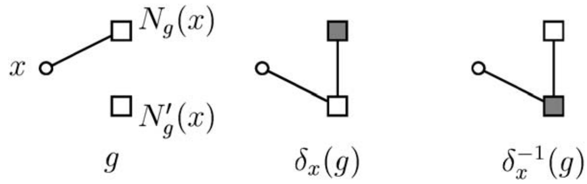

Fig. 8. The diagrams for g, $\delta_x(g)$ and $\delta_x^{-1}(g) = C\lambda_x(g)$.

Proof. First of all, $\delta_z\delta_x^{-1} = \lambda_z CC\lambda_x = \lambda_z\lambda_x$, and since x is isolated in g, we have $\lambda_x(g) = g$, and the first claim follows. For the second claim, we observe that $\delta_x^2 = \lambda_x(C\lambda_x C)$. By Lemma 9, x is isolated in $C\lambda_x C(g)$ and $C\lambda_x C(g) - x = C(g - x)$. It follows that x is isolated in $\delta_x^2(g)$, and $\delta_x^2(g) - x = \lambda_x C\lambda_x C(g) - x = C\lambda_x C(g) - x = C(g - x)$. □

Theorem 14. *The group $\langle \Delta \rangle_D$ generated by the operations $\delta_x = \lambda_x C$ is transitive on $\mathscr{G}_D$.*

Proof. The proof is by induction on the number of vertices. If $|D| \leqslant 2$, then again the claim is clear. Assume now that the claim holds for sets of vertices of cardinality $n - 1$, and let $|D| = n$. Let $g \in \mathscr{G}_D$. We show that there exists an operation $\varphi \in \langle \Delta \rangle_D$ such that $\varphi(g) = \mathbf{0}_D$.

We first show that there exists $\beta(g)$, for some $\beta \in \langle \Delta \rangle_D$, such that $\beta(g)$ contains an isolated vertex. Let $x \in D$, and let $h = g - x$. By the induction hypothesis, there exists an operation $\gamma \in \langle \Delta \rangle_D$ such that $\gamma(h) = \mathbf{0}_{D\setminus\{x\}}$. Clearly, $\gamma(h) = \gamma(g) - x$. If x is not universal in $\gamma(g)$, then the vertices $y \notin N_{\gamma(h)}(x)$ are isolated in $\gamma(g)$, and, if x is universal in $\gamma(g)$, then x is isolated in $\delta_x\gamma(g)$ (which is equal to $C\gamma(g)$ in this case).

We can now assume that x is isolated already in g. By Theorem 10, there exists a composition $\gamma \in \langle C, \Lambda \rangle_D$ of complementations and local complementations such that $\gamma(g) = \mathbf{0}_D$. By Lemma 13, $\alpha(g) = \mathbf{0}_D$, where α is obtained from γ by replacing each occurrence of C by δ_x^2, and each occurrence of λ_z by $\delta_z\delta_x^{-1}$. □

Corollary 15. *Every graph can be constructed from the discrete graphs by using the operations δ_x for vertices x.*

5. Switching with local complementation

We show now that the group $\langle \Sigma, \Lambda \rangle_D$, generated by switchings and local complementations, is transitive. Indeed, this follows from the proof of Theorem 14, because if $g = (D, E)$ is a graph and $x \in D$, then the vertex x is isolated in $\sigma_{N_g(x)}(g)$, and moreover, it can be shown that $\delta_y^2 = (\lambda_y\sigma_y)^2$ for all $y \in D$. However, we shall prove a stronger result in Theorems 16 and 17.

In order to prove that $\langle \Sigma, \Lambda \rangle_D$ is transitive it suffices to show, by Theorem 10, that the complement operation C can be expressed in terms of switchings and local complementations. In the following theorem, it is proved that C belongs to a subgroup

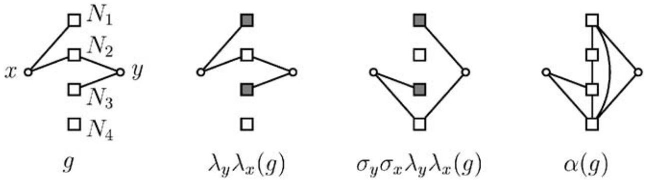

Fig. 9. The diagrams for g, $\lambda_y\lambda_x(g)$, $\sigma_y\sigma_x\lambda_y\lambda_x(g)$ and $\alpha(g)$.

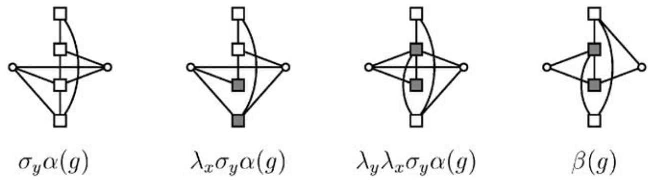

Fig. 10. The diagrams for $\sigma_y\alpha(g)$, $\lambda_x\sigma_y\alpha(g)$, $\lambda_y\lambda_x\sigma_y(g)$ and $\beta(g)$.

of $\langle \Sigma, \Lambda\rangle_D$ generated by four elements λ_x, λ_y, σ_x, and σ_y determined by any two vertices x and y.

Theorem 16. *Let* $x, y \in D$ with $x \neq y$. *Then*

$$C = \lambda_x\lambda_y\sigma_x\sigma_y\lambda_y\lambda_x\sigma_y\lambda_x\lambda_y\sigma_y\sigma_x\lambda_y\lambda_x. \tag{7}$$

In particular, $C \in \langle \Sigma, \Lambda\rangle_D$.

Proof. Let φ denote the right-hand side of (7). The expression for φ is palindromic, which implies that $\varphi^{-1} = \varphi$. In fact, $\varphi = \alpha^{-1}\sigma_y\alpha$, where

$$\alpha = \lambda_x\lambda_y\sigma_y\sigma_x\lambda_y\lambda_x \tag{8}$$

satisfies $\alpha^{-1} = \alpha$, since $\sigma_x\sigma_y = \sigma_y\sigma_x$. Therefore, if can show that $\varphi(g) = C(g)$ for a graph g, where xy is not an edge, then

$$\varphi(C(g)) = \varphi^{-1}(C(g)) = \varphi^{-1}(\varphi(g)) = g = C(C(g))$$

and thus also the complement of g (where xy is an edge) satisfies the claim. Hence, without loss of generality, we can assume that in g we have $xy \notin E$.

We choose (see Fig. 9) the diagram (A, S) for g, where $A = \{x, y\}$ and S consists of the sets

$$N_1 = N_g(x) \cap N_g'(y), \qquad N_2 = N_g(x) \cap N_g(y),$$

$$N_3 = N_g'(x) \cap N_g(y), \qquad N_4 = N_g'(x) \cap N_g'(y).$$

In Fig. 9, we have derived the diagram for $\alpha(g)$ (using two steps between the diagrams). Similarly, in Fig. 10 we have derived the diagram for $\beta(g)$, where $\beta = \sigma_y\lambda_y\lambda_x\sigma_y\alpha$. Finally, in Fig. 11 the graph $\varphi(g)$ is derived, and one sees from the result that $C(g) = \varphi(C)$, as required. $\square$

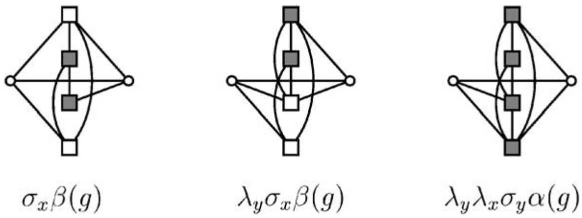

Fig. 11. The diagrams for $\sigma_x\beta(g)$, $\lambda_y\sigma_x\beta(g)$, and $C(g)=\lambda_x\lambda_y\sigma_y\beta(g)$.

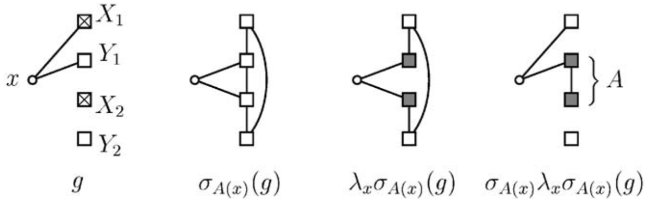

Fig. 12. $g\oplus A=\sigma_{A(x)}\lambda_x\sigma_{A(x)}(g)$, where $A=X_2\cup Y_1$ and $A(x)=X_1\cup X_2$.

The following result is a corollary to Theorems 10 and 16.

Theorem 17. *The group $\langle\Sigma,\Lambda\rangle_D$ is transitive on $\mathscr{G}_D$. In fact, for all $x,y\in D$ with $x\neq y$, $\langle\sigma_x,\sigma_y,\Lambda\rangle_D$ is transitive on $\mathscr{G}_D$.*

In Theorem 17, we do need two switchings in addition to the local complementations. Indeed, in the set $D=\{x,y,z\}$ of vertices, the graph with an isolated vertex z and one edge xy cannot be reduced to the discrete graph by using operations from $\langle\sigma_x,\Lambda\rangle_D$.

We continue by showing that in addition to the switchings local complementations λ_x are needed only with respect to one fixed vertex x such that $\langle\Sigma,\lambda_x\rangle_D$ is transitive group.

Let $x\in D$ and $A\subseteq D$, and let

$$A(x)=N_g(x)+A.$$

For Lemma 18, we observe that, for all $B\subseteq D$, if $x\notin B$, then $N_{\sigma_B(g)}(x)=N_g(x)+B$.

Lemma 18. *Let g be a graph on D, $x\in D$ and $A\subseteq D\setminus\{x\}$. Then $g\oplus A=\sigma_{A(x)}\lambda_x\sigma_{A(x)}(g)$.*

Proof. In Fig. 12 we have the diagram for g by the sets $\{x\}$ and the set S consisting of the subsets $X_1=N_g(x)\setminus A$, $Y_1=N_g(x)\cap A$, $X_2=N'_g(x)\cap A$, $Y_2=N'_g(x)\setminus A$. Thus, $A=Y_1\cup X_2$ and $A(x)=X_1\cup X_2$ (marked with crosses in Fig. 12). From the result, we can then read the claim $\sigma_{A(x)}\lambda_x\sigma_{A(x)}(g)=g\oplus A$. $\square$

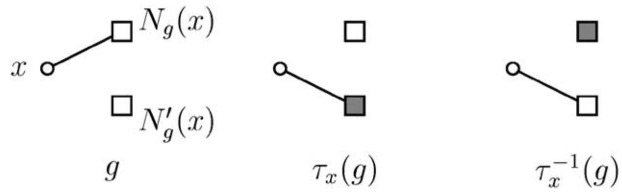

Fig. 13. The diagrams for $\tau_x(g) = \lambda_x\sigma_x(g)$ and $\tau_x^{-1}(g) = \sigma_x\lambda_x(g)$.

We prove now that the local complementations needed can be restricted to one fixed vertex.

Theorem 19. *Let $x \in D$ be a fixed vertex. Then the group $\langle \Sigma, \lambda_x \rangle_D$ is transitive on $\mathscr{G}_D$.*

Proof. Let $g \in \mathscr{G}_D$, and let $h = \sigma_{N'_g(x)}(g)$. Then the vertex x is universal in h, that is, $N_h(x) = D \setminus \{x\}$. Assume that $e_1, e_2, \ldots, e_k$ are the edges in $h - x$. Define inductively, $h_0 = h$ and $h_{i+1} = h_i \oplus e_i$ for $i = 0, 1, \ldots, k-1$. It is clear that $h_k - x = \mathbf{0}_{D\setminus\{x\}}$, since $h_i \oplus e_i$ simply removes the edge e_i from h_i. Therefore, h_k is a star graph, where x remains to be universal, and hence $\sigma_x(h_k) = \mathbf{0}_D$. By Lemma 18, $h_{i+1} = \sigma_{e_i(x)}\lambda_x\sigma_{e_i(x)}(h_i)$ for each i, and thus

$$\sigma_x\sigma_{e_k(x)}\lambda_x\sigma_{e_k(x)} \ldots \sigma_{e_1(x)}\lambda_x\sigma_{e_1(x)}\sigma_{N'_g(x)}(g) = \mathbf{0}_D,$$

which proves the claim. $\square$

As a corollary, we have:

Corollary 20. *Let $x \in D$ be a fixed vertex. Then every graph g on D can be constructed from the discrete graph by using switchings σ_y $(y \in D)$ and the local complementation λ_x with respect to x.*

For a vertex $x \in D$, let

$$\tau_x = \lambda_x\sigma_x,$$

see Fig. 13. We observe that $\tau_x^4 = \imath$, and thus that $\tau_x^{-1} = \tau_x^3 = \sigma_x\lambda_x$ for all $x \in D$. Let $\Theta = \{\tau_x \mid x \in D\}$.

Theorem 21. *The operations $\tau_x = \lambda_x\sigma_x$ $(x \in D)$ generate a transitive subgroup on $\mathscr{G}_D$.*

Proof. The claim for $|D| \leqslant 2$ is obvious. Assume thus that $|D| \geqslant 3$, and let $g = (D, E)$ be a graph. We show by induction that g is Θ-equivalent to the discrete graph on D.

Let $x \in D$ be a fixed vertex. By the induction hypothesis, there exists an operation $\alpha \in \Theta$ such that $\alpha(g - x) = \mathbf{0}_{D\setminus\{x\}}$. Denote $h = \alpha(g)$, for short. In the graph h, the subgraph $h - x$ is discrete. If $N_h(x) = \emptyset$, then clearly $h = \mathbf{0}_D$, and if $N_h(x) = D \setminus \{x\}$, then $\tau_x(h) = \sigma_x(h) = \mathbf{0}_D$. Suppose then that $N_h(x) \neq \emptyset$ and let $y \in N_h(x)$ be a fixed vertex. Let $z \in N'_h(x)$, and define an operation $\beta_z \in \Theta$ as follows:

$$\beta_z = \tau_z\tau_y\tau_z^{-1}\tau_y^{-1}.$$

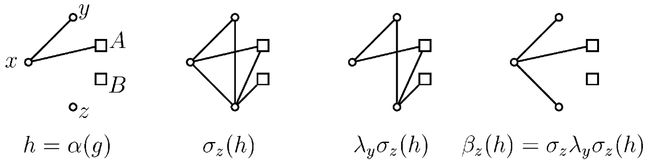

Fig. 14. The graphs in the proof of Theorem 21.

(Note that this operation is the commutator of the elements τ_z and τ_y.) We show that $\beta_z(h) = h \oplus \{x,z\}$. The claim follows from this, since by applying the operations β_z for all $z \in N'_h(x)$ we obtain the graph h_1 where the vertex x is universal, and the rest of the graph is discrete. Then, as in the above, $\tau_x(h_1) = \sigma_x(h_1) = \mathbf{0}_D$.

The valency of the vertex y equals one in h (that is, $|N_h(y)|=1$), and hence $\tau_y^{-1}(h) = \sigma_y(h)$, since $\tau_y^{-1} = \sigma_y\lambda_y$ and $\lambda_y(h) = h$. Similarly, in $\lambda_y(h)$ the valency of the vertex z is equal to one, and hence $\tau_z^{-1}\tau_y^{-1}(h) = \sigma_z\sigma_y(h)$. Therefore, $\beta_z(h) = \tau_z\lambda_y\sigma_y\sigma_z\sigma_y(h) = \tau_z\lambda_y\sigma_z(h)$, since $\sigma_y\sigma_z = \sigma_z\sigma_y$ and $\sigma_y^2 = \imath$. Also, the valency of z is one in $\sigma_z\lambda_y\sigma_z(h)$, see Fig. 14, where $A = N_h(x) \setminus \{y\}$ and $B = N'_h(x) \setminus \{z\}$. Therefore, $\beta_z(h) = \sigma_z\lambda_y\sigma_z(h)$. Now, $\beta_z(h) = h \oplus \{x,z\}$ follows, see Fig. 14. $\square$

We have then the following corollary.

Corollary 22. *All graphs can be constructed from the discrete graphs by using the operations τ_x for vertices x.*

6. On the number of operations

In this section, we estimate the number of operations needed to construct a graph from the discrete graph. We shall concentrate on the full set $\Gamma = \Sigma \cup \Lambda \cup \{C\}$ of generators.

Let $M(g)$ denote the minimum number of operations α_i from Γ such that $\alpha(\mathbf{0}_D) = g$ for $\alpha = \alpha_{M(g)} \ldots \alpha_1$. Moreover, let

$$M(n) = \max\{M(g) \mid g = (D,E),\ n = |D|\}.$$

We shall first give a lower bound for M. Fix a set D of vertices with $n = |D|$. Since $\alpha^2 = \imath$ for all $\alpha \in \Gamma$, there are at most $(2n+1)(2n)^{i-1}$ operations in $\langle \Gamma \rangle_D$ that are compositions of i operations of the generators. Therefore, we must have

$$2^{n(n-1)/2} \leqslant 1 + (2n+1)\sum_{i=1}^{M(n)} (2n)^{i-1} \leqslant 1 + (2n+1)\frac{(2n)^{M(n)} - 1}{2n-1}, \tag{9}$$

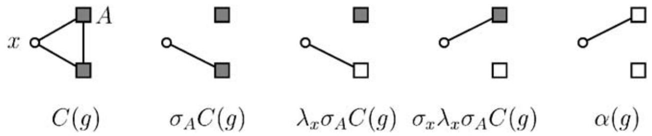

Fig. 15. The derivation of $\alpha(g)$ for $\alpha = \eta_x \sigma_A C$.

where the left-hand side is the number of all graphs on D. Inequality (9) implies

$$(2n)^{M(n)} \geqslant \frac{\left(2^{n(n-1)/2}-1\right)(2n-1)+(2n+1)}{2n+1} > \frac{2^{n(n-1)/2}(2n-1)}{2n+1}$$

and taking logarithms in the base 2 of both sides, we obtain

$$M(n) > \frac{n(n-1)}{2\log(2n)} - \frac{\log(2n+1)-\log(2n-1)}{\log(2n)}$$

and therefore, as easily seen,

$$M(n) > \frac{n(n-1)}{2(\log n + 1)} - 1. \tag{10}$$

The lower bound in (10) is not optimal, since, as we have seen, the group $\langle \Gamma \rangle_D$ satisfies several identities.

We now turn to an upper bound for $M(n)$. We begin with a lemma that can be used to give an alternative proof for the transitivity of the group $\langle \Gamma \rangle_D$.

Recall that $\eta_x = \lambda_x \sigma_x \lambda_x$. In Lemma 23, it is shown that, for an isolated vertex x and a subset A, the operation $\eta_x \sigma_A C$ adds the edges xy with $y \in A$ to the graph, and leaves the rest of the graph unchanged. For the proof of this result, we refer to Fig. 15.

Lemma 23. *Let $g = (D, E)$ be a graph with an isolated vertex let x. Denote $h = g - x$. Let $A \subseteq D \setminus \{x\}$, and $\alpha = \eta_x \sigma_A C$. Then $N_{\alpha(g)}(x) = A$, and $\alpha(h) = h$.*

By Lemma 8, we have

$$\eta_x \sigma_A C = C \eta_x \sigma_A \tag{11}$$

since $\sigma_A C = C \sigma_A$. In particular, for each $x, y \in D$ and $A \subseteq D \setminus \{x\}$, $B \subseteq D \setminus \{y\}$, we have

$$\eta_x \sigma_A C \eta_y \sigma_B C = \eta_x \sigma_A \eta_y \sigma_B,$$

that is, the complementations cancel each other.

Theorem 24. *For each $n \geqslant 3$ and $g = (D, E)$ with $|D| = n$,*

$$M(g) \leqslant |E| + 3(n-2)$$

and

$$M(n) \leqslant \frac{n(n-1)}{4} + 3(n-2).$$

Proof. If the graph g is complete, then $\sigma_x \lambda_x(g) = \mathbf{0}_D$, and hence $M(g) = 2$. We now assume that g is neither discrete nor complete, and thus that there are two vertices, x_1 and x_2, such that $x_1x_2 \notin E$. We can also assume that there exists a vertex $x_3 \in D$ such that $x_1x_3 \notin E$ or $x_2x_3 \notin E$. For, otherwise, the vertices x_1 and x_2 are isolated in $\sigma_{x_2}\sigma_{x_1}(g)$, which has $|E| - 2(n-2)$ edges. Therefore, if the claim holds for $\sigma_{x_2}\sigma_{x_1}(g)$, it holds for g.

Let $D = \{x_1, x_2, \ldots, x_n\}$, and denote $g_i = g[\{x_1, \ldots, x_i\}]$, and $d_i = |N_{g_i}(x_i)|$ for $i = 1, 2, \ldots, n$. Clearly, $\sum_{i=1}^{n} d_i = |E|$, since each edge is counted exactly once in the summation. Notice that $d_1 = 0$, and also $d_2 = 0$, since $x_1x_2 \notin E$. We apply Lemma 23 to the instances x_i and $A_i = N_{g_i}(x_i)$, in the order $i = 3, \ldots, n$. Using (11) to reduce complementations, we obtain an operation α such that $\alpha(\mathbf{0}_D) = g$. Here

$$\alpha = \gamma \cdot \eta_{x_n}\sigma_{A_n} \cdot \eta_{x_{n-1}}\sigma_{A_{n-1}} \ldots \eta_{x_3}\sigma_{A_3},$$

where $\gamma = \imath$ if n is even, and $\gamma = C$ if n is odd. Moreover, $\eta_{x_3}\sigma_A(\mathbf{0}_D) = \sigma_{x_3}\lambda_{x_3}\sigma_{A_3}(\mathbf{0}_D)$, since $x_1x_3 \notin E$ or $x_2x_3 \notin E$, that is, $|A_3| \leqslant 1$ (and the last λ_{x_3} is unnecessary, see Fig. 15). Therefore,

$$\begin{aligned} M(g) &\leqslant \sum_{i=3}^{n} (d_i + 3) - 1 + p(n) = |E| + 3(n-2) - 1 + p(n) \\ &= |E| + 3n - 7 + p(n), \end{aligned}$$

where $p(n) = 0$, if n is even, and $p(n) = 1$, if n is odd. This shows the first claim.

We observe that $M(g) \leqslant M(C(g)) + 1$, and, by the above, $M(C(g)) \leqslant |E| + 3n - 7 + (1 - p(n))$, since if n is odd, then α ends with complementation. The second claim follows, since either g or $C(g)$ has at most $n(n-1)/4$ edges, and $M(C(g)) \leqslant (n(n-1)/2 - |E|) + 3n - 7 + p(n)$ implies, by the above, that $M(g) \leqslant (n(n-1)/2 - |E|) + 3(n-2)$. $\square$

The bound given in Theorem 24 is not optimal. Indeed, one can show for the small values that $M(3) = 2$, $M(4) = 4$ and $M(5) = 7$ while Theorem 24 gives $M(3) \leqslant 4$, $M(4) \leqslant 9$ and $M(5) \leqslant 14$.

As a conclusion we summarize the results of this section in the following corollary.

Corollary 25. *For each $n \geqslant 3$ and $g = (D, E)$ with $|D| = n$, we have*

$$\frac{n(n-1)}{2(\log n + 1)} - 1 < M(n) \leqslant \frac{n(n-1)}{4} + 3(n-2).$$

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